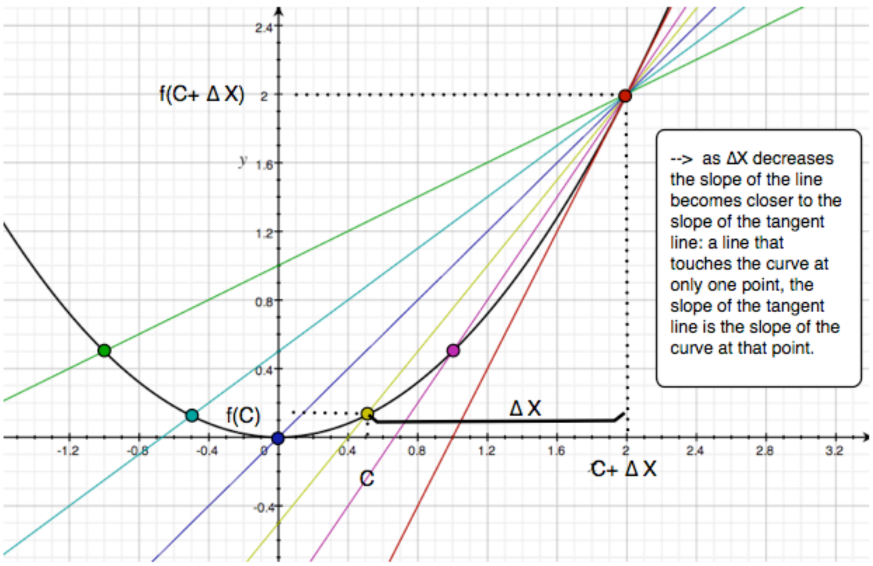
DERIVATIVES:

The concept of a Derivative:

-> The rate of change of a function with respect to an independent variable. Or in simple words and most cases: the slope.

Two applications: Tangent Lines, Instantaneous rate of change.

Graphically:



pg. 96

Slope of Tangent Line$=\dfrac{\Delta y}{\Delta x}=\dfrac{f(c+\Delta x)-f(c)}{\Delta x}$, where Δx is as small as possible, essentially equal to zero, so:

$$\boxed{\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=f'(c)=\text{the slope of the Tangent Line/the Derivative at the point.}}$$

Also, $\boxed{\lim_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)}$

Note: $\dfrac{f(c+\Delta x)-f(c)}{\Delta x}$ can also be simplified to $\dfrac{f(c+\Delta x)-f(c-\Delta x)}{2\Delta x}$, if it's easier to remember…

Example:

Find the derivative of x^3+2x

$$\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3+2(x+\Delta x)-(x^3+2x)}{\Delta x}$$

$$\lim_{\Delta x\to 0}\frac{x^3+3x^2\Delta x+3x9(\Delta x)^2+(\Delta x)^3+2x+2\Delta x-x^3-2x}{\Delta x}$$

$$\lim_{\Delta x\to 0}3x^2+3x\Delta x+(\Delta x)^2+2=3x^2+2$$

Numerically:

As Δx approaches zero in the above graph, the slopes approach the slope of the tangent line, leveling off slowly. Slopes can be estimated from tables of values and the number they approach is the slope of the tangent line, the derivative.

Analytically:
Finding the derivative algebraically (normally).

pg 102
235

<u>Tangent Lines:</u>

->A line touching a curve at only one point and therefore having the slope of the curve at that point. To find its equation: use point slope form (slope from the derivative at the point).

OR: Tangent Line approximation:

From point slope form, the equation of a tangent line at the point (c, f(c)):

$$\boxed{y = f(c) + f'(c)(x - c)}$$

Example:
Find the equation of the line tangent to $y = 8x^3 - 5x^2$ at the point (3,171).
Therefore $y' = 24x^2 - 10x$.
and thus y'(3)=186. Using point-slope form we determine that the equation of the tangent line at (3,171) is y - 171=186(x - 3), or in slope intercept form, y=186x – 387

OR: $y = f(c) + f'(c)(x - c)$
and $y = 171 + 186(x - 3)$. Simple.

pg 104

<u>Instantaneous rate of change:</u>

Average velocity= $\dfrac{\text{change in position}}{\text{change in time}} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(t + \Delta t) - s(t)}{\Delta t}$

However, $\boxed{\textbf{INSTANT velocity=} \lim\limits_{\Delta t \to 0} \dfrac{s(t + \Delta t) - s(t)}{\Delta t} =}$ as the change in time gets closer to zero the velocity becomes defined at an *instant*.

Conditions of Differentiability:

For a function to be differentiated the graph must be continuous **and** not have any corners (sharp turns). ALL differentiable functions are continuous, but not all continuous functions are differentiable.

Also, if the tangent line is vertical then the slope is undefined (infinite).

Slope of a curve at a point:

-> The derivative evaluated at that point.

OR: For finding the derivative at a point (c, f(c)) an alternate form of the limit definition can be

used: $\boxed{\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \text{ (where the limits from both directions exist and are equal)}}$

Examples:

1. Consider:
$y(x) = 4x^2 + 3x - 4$, We want to know the slope of the function at the point (2,18). The first derivative is:
$y'(x) = 8x + 3$, Plug 2 in for x to get:
$y'(2) = 19$, So the slope of at the point (2,18) is 19.

2. If the slope at the point of the original function is zero (the line tangent is horizontal), you will get zero as the derivative. Consider the equation in the first example:
$y(x) = 4x^2 + 3x - 4$, The derivative, we determined, is
$y'(x) = 8x + 3$, But now we want to know the slope at the point (-3/8, -73/16). So we plug -3/8 in for x to get y'(-3/8)=8(-3/8) + 3 which is y'(-3/8)=0. So the slope of at the point (-3/8, -73/16) is 0 and the line tangent to the graph at that point is a horizontal line.

3. Another thing to consider is a vertical tangent. Consider the equation
$y(x) = x^{1/3}$, We want to know the derivative at the point (0,0). We know:
$y'(x) = \frac{1}{3}x^{-1/3}$, Upon plugging in x=0 we discover
$y'(0) = \frac{1}{0}$, or undefined. What is the slope of a vertical line? undefined. That means that there is a vertical tangent at the point (0,0) for the equation $y(x) = x^{1/3}$.

NOTE: This only applies to x values that fall into the domain of the original function. If the y-value for an x-value is undefined in the original function, then there is no derivative at that point. This is shown in the next example.

4. One final note is that y(x) has to be differentiable at x in order for it to have a slope at that point. If y(x) is discontinuous or has a sharp turn, there is no slope at that point. Examples are y(x)=|x| at the point (0,0) and y(x)=1/x at the point (0,0). When examining y(x)=|x|, we find it has a sharp turn at x=0 and the derivative is y'(x)=-1,(-infinity,0) and 1,(0,infinity). Thus x approaches two different y values as it approaches 0. And for the graph:
$y(x) = \frac{1}{x}$, it is undefined at x=0 (0 is not in the domain), so no derivative is possible.

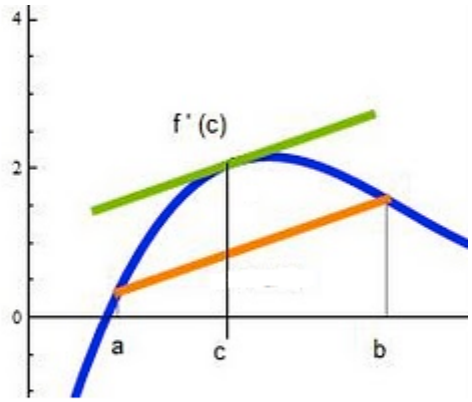
Approximating rate of change from sets of values:
Rate of change can be estimated using tables of values by simply calculating the slope for the interval around that point. For example, if given the following set of values:
P1:(1,12) P2:(3,15) P3:(5,20) P4:(7,10)
and asked to estimate the slope of the curve at x=2, you would simply calculate the slope from Point 1 to Point 2. These values could also be presented in a table or estimated from a graph.

Derivative as a Function

Ch - 3.2 The Mean Value Theorem

If f is continuous on the closed interval [a,b] and differentiable on the open interval (a,b) then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



IN OTHER WORDS ... If there is a function that is continuous on the closed interval between two points and differentiable on the open interval between the same two points, there is a third point in that interval with the same slope as the slope between both endpoints.

Example

Given $f(x) = x^2 + 5$, find all the values of c in the open interval (-1,4) such that $f'(c) = \dfrac{f(4) - f(-1)}{4 - -1}$

Process

We can evaluate the given difference quotient to determine the slope of the secant line between x = 1, 4.

$$\frac{f(4) - f(-1)}{4 - -1} = \frac{15}{5} = 3$$

If $f'(x) = 2x$, by the Mean Value Theorem, we know that $3 = 2c$.

Solution

$c = \dfrac{3}{2}$

* for more practice do p176 # 11-24

Ch - 6.2 Equations Involving Derivatives

A **differential equation** is simply an equation involving a derivative. The solution to a differential equation is in the form of a function (y = f(x)) that satisfies the equation "when y and its derivatives are replaced by f(x) and its derivatives". A general solution uses the letter C to represent any real number, while a particular solution yields a specific answer evaluated at a point. To effectively work with diff eq's, one must be able to translate between the verbal and numerical versions of the equation and vice versa.

For example ...

- The rate of change of L with respect to i is inversely proportional to i + 17: $L'(j) = \frac{k}{j+17}$
- The rate of change of M with respect to w is proportional to the square of w: $M'(w) = k(w^2)$
- $y'(x) = \frac{k}{120x}$: The rate of change of y with respect to x is inversely proportional to 120 times x
- $T'(p) = \frac{k}{p(T-58)}$: The rate of change of T with respect to p varies jointly as 5p and T-58

*for more practice do p418 # 11-14

Ch - 3.3 Relationship between the behavior of f and the sign of f'

When f is increasing (or has a positive slope), f' is greater than zero (> 0)
When f is decreasing (or has a negative slope), f' is less than zero (< 0)
When f has a slope of zero, f' has a value of zero and crosses the x-axis (=0)
When f has an undefined slope at a point, f' at that point does not exist

Example

Find the open intervals on which $f(x) = x^2$ is increasing or decreasing

Process

1. To determine the critical points (points where the slope of f(x) may change *occur where f'(x) equals zero or is undefined*) set f'(x) = 0. Since f'(x) = 2x and f'(x) is defined for all values of x, the only critical point occurs at x = 0.
2. Based on the obtained critical points, set up intervals in which f(x) will have the same slope behavior. Evaluate the sign of the slope at test points within those intervals.

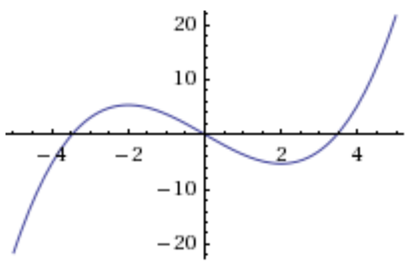
Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	x = -1	x = 1
Sign of f'(x)	-	+
Conclusion	Decreasing	Increasing

Solution

f(x) is decreasing on the interval $(-\infty,0)$ and increasing on the interval $(0, \infty)$

Ch - 3.3 Corresponding Characteristics of f(x) and f'(x)

$f(x) = \frac{1}{3}x^3 - 4x$

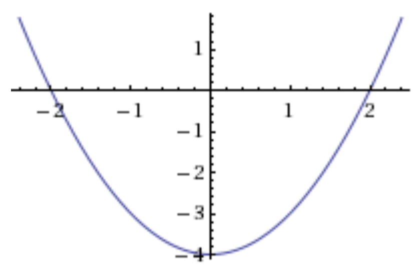


From this graph we deduce that f'(x) …

- Is positive from x = 2 to infinity
- Is positive from x = -2 to negative infinity
- Is negative between (-2,2)
- Accelerates while increasing (increases at an increasing rate)
- Crosses the x-axis at the points x = 2, -2

Conversely,

$f'(x) = x^2 - 4$



From this graph we deduce that f(x)…

- Is decreasing in the interval (-2,2)
- Is increasing from negative infinity to x = -2
- Is increasing from x = 2 to positive infinity
- Accelerates while increasing
- Has a relative minimum at x = 2
- Has a relative maximum at x = -2

Ch - 3.1 ***Side Note

Relative Extrema

- f(c) is the relative minimum of f on the interval I if f(c) is less than or equal to f evaluated at all values of x contained in I.
- In the same way, f(c) is the relative maximum of f on the interval I if f(c) is greater than or equal to f evaluated at all values of x contained in I.
- Can only occur at critical points (when f'(c)=0 or when f' is undefined at x=c)

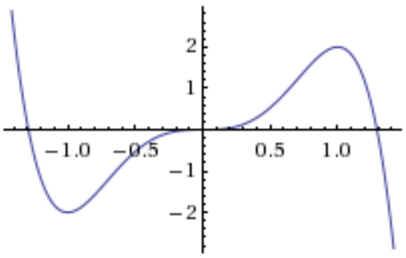
SO… The graph of f can only have a turning point when f' evaluated at that point is equal to zero.

Second Derivative

Ch - 3.4 Characteristics of the graphs of f(x), f’(x) and f’’(x)

*since we’ve already discussed the relationship between the graphs of f(x) and f’(x), this page is primarily focused on comparing both graphs to that of f’’(x) and vice versa

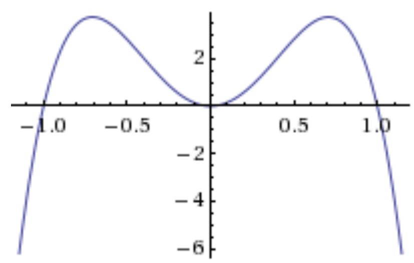
f(x) = $3x^5 + 5x^3$



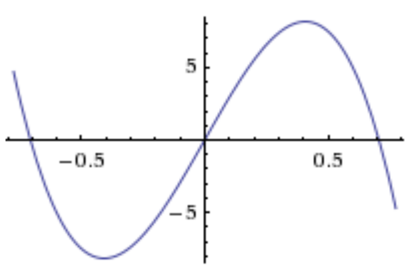
- Since the concavity of f(x) is positive in the intervals (-∞,≈-.75), and (0,≈..75), the graph of f’’(x) must have positive values in the same intervals
- Since the concavity of f(x) is negative in the intervals (≈-.75,0) and (≈.75,∞), the graph of f’’(x) must have negative values in the same intervals
- Since the graph flattens out at x = 0, f’’(0) is also equal to zero

- Since the graph has positive slopes in the intervals (-∞,≈-.75), and (0,≈.75), the graph of f’’(x) must have positive values in the same intervals
- Since the graph has negative slopes in the intervals (≈-.75,0) and (≈.75,∞), the graph of f’’(x) must have negative values in the same intervals
- Since f’(x) has relative maximums at ≈-.75 and ≈ .75, the graph of f’’(x) crosses the x-axis in the negative direction at those points
- Since f’(x) has a relative maximum at x = 0, the graph of f’’(x) crosses the x-axis in the positive direction at that point

f’(x)= $-15x^4 + 15x^2$



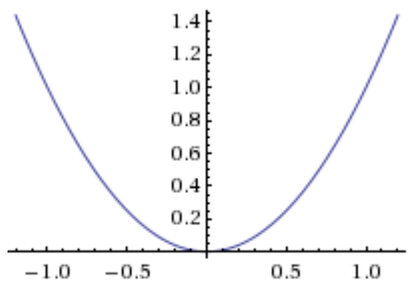
f’’(x) = $-60x^3 + 30x$



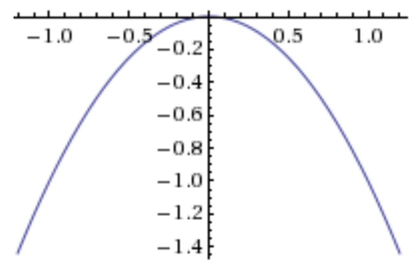
- Since the graph is positive in the intervals (-∞,≈-.75), and (0,≈.75), the graph of f’(x) must have a positive slope and the graph of f(x) must have positive concavity in the same intervals
- Since the graph is negative in the intervals (≈-.75,0) and (≈.75,∞), the graph of f’(x) must have a negative slope and the graph of f(x) must have negative concavity in the same intervals
- Since the graph crosses the x-axis at x= ≈-.75, 0 and ≈ .75, the graph of f’(x) has no slope at those points

Relationship between the concavity of f(x) and the sign of Ch - 3.4 f’’(x)

Concavity is used to explain the behavior or curve of a slope. A graph that is concave up is increasing or decreasing at an increasing rate over time while a graph that is concave down is increasing or decreasing at a decreasing rate over time. Graphically, we say that if graphs were bowls, a graph with **positive concavity catches water** while one with **negative concavity causes water to run off** it.



catches water



water runs off

If f’’(x) is ***positive,*** then f’(x) has a ***positive slope*** and f(x) has ***positive concavity***
If f’’(x) is ***negative,*** then f’(x) has a ***negative slope*** and f(x) has ***negative concavity***

Example

Determine the open intervals on which the graph of f(x) = $\frac{1}{12}x^4 - 2x^2$ is concave up and concave down

Process

1. To determine the points at which the concavity of f(x) might change, find all PIP’s. Since f’’(x) = x2-4, the PIP’s occur at x= 2 and x = -2. Since f’’(x) exists for all values of x, those are the only PIP’s.
2. Set up a table as shown in the previous example

Interval	-∞ < x < -2	-2 < x < 2	2 < x < ∞
Test Value	x = -3	x = 0	x = 3
Sign of f’’(x)	+	-	+
Conclusion	concave up	concave down	concave up

Solution

f(x) has positive concavity in the intervals (-∞, -2) and (2, ∞) and negative concavity in the interval (-2, 2).
* for more practice do p195#1-10

Ch - 3.4 Points of Inflection

A point of inflection is a point on a graph at which the concavity changes and at which the tangent line to that point exists. To locate potential inflection points (or PIP's), determine for which values of x $f'(x)$ will equal zero or not exist.

Example (p 193 in Calc book)

Locate the points of inflection for the graph of $f(x) = x^4-4x^3$ and discuss the concavity of the function.

Process

1) If $f(x) = x^4-4x^3$ then,
   $f'(x) = 4x^3-12x^2$ and,
   $f'(x) = 12x^2-24x$
2) $f'(x) = 0$ when $x = 0$ or 2 and $f(x)$ is defined for all values of x. Therefore, $x=0$ and $x=2$ are the only PIP's for this graph

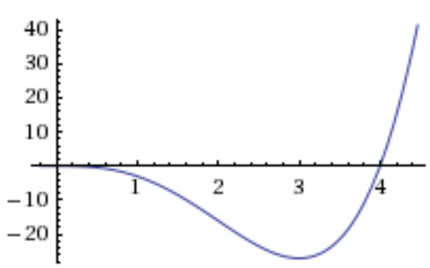
3) Set up intervals based on the determined PIP's, and evaluate the second derivative at test values chosen within those intervals. If the second derivative attained is positive, the graph is concave up within that interval. If the second derivative is negative, the graph is concave down within that interval. If the concavity changes at the PIP, it is confirmed as an inflection point. In this example, both $x = 0$ and $x = 2$ prove to be inflection points.

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	+	-	+
Conclusion	concave up	concave down	concave up

Solution

The inflection points occur at $x = 0$ and $x = 2$. In the interval $(-\infty, 0)$ and $(2, \infty)$ the graph is concave upward. In the interval $(0, 2)$ the graph is concave downward. Our results coincide with the graph pictured below.

$f(x) = x^4 - 4x^3$



Book	Computation of Derivatives	Examples
2.2	**I. Basic Rules**	
	A. Constant Rule: $\frac{d}{dx}[c] = 0$ – Derivative of a constant is always zero.	$\frac{d}{dx}(5) = 0$
	B. Constant Multiple Rule: $\frac{d}{dx}[cf(x)]=f'(x)$ – Derivative of a constant times the function is constant times the derivative of the function.	$\frac{d}{dX}(3X^2) = 6X$ $= 3\left(\frac{d}{dX}(X^2)\right) = 3(2X)$
	C. Power Rule: $\frac{d}{dx}[x^n]=nx^{n-1}$ – Derivative of X raised to an n power is n times x to the n minus 1 power.	$\frac{d}{dx}(x^5) = 5x^4$
	D. Sum Rule: $\frac{d}{dx}[f(x)+g(x)] = f'(x) + g'(x)$ – Derivative of the sum of two functions is the derivative of the first function plus the derivative of the second function.	$\frac{d}{dx}(x^2+2X) = 2x+2$ $= \frac{d}{dx}(x^2) + 2\left(\frac{d}{dx}(X)\right)$
	E. Difference Rule: $\frac{d}{dx}[f(x)–g(x)]= f'(x)$-g'(x) – Derivative of the difference of two functions is the derivative of the first function minus the derivative of the second function	$\frac{d}{dX}(4X^2-6X) = 8X-6$ $= 4\left(\frac{d}{dX}(X^2)\right) - 6\left(\frac{d}{dX}(X)\right)$
2.3	**II. Product and Quotient Rules**	
	A. Product Rule: $\frac{d}{dx}[f(x)g(x)]=f'(x)g(x)+f(x)g'(x)$ – Derivative of the product of two functions is the derivative of the first times the second plus the first times the derivative of the second.	$\frac{d}{dX}((2X^3-4)(X^2+X)) = 10X^4+8X^3-8X-4$ $= (X^2+X)\left(\frac{d}{dX}(2X^3-4)\right) + (2X^3-4)\left(\frac{d}{dX}(X^2+X)\right)$ $= (2X+1)(2X^3-4) + 2(X^2+X)(3X^2)$
	B. Quotient Rule: $\frac{d}{dx}[f(x)/g(x)]=[f'(x)g(x)-f(x)g'(x)]/[g(x)]^2$ – Derivative of the quotient of two functions is the derivative of the top times the bottom minus the top times the derivative of the bottom all over the bottom squared.	$\frac{d}{dx}\left(\frac{4x}{5x^2+3}\right) = -\frac{4(5x^2-3)}{(5x^2+3)^2}$ $= 4\frac{(5x^2+3)\left(\frac{d}{dx}(x)\right) - x\left(\frac{d}{dx}(5x^2+3)\right)}{(5x^2+3)^2}$ $= \frac{4(5x^2-5(2x)x+3)}{(5x^2+3)^2}$
2.4	**III. Chain Rule**	
	A. Chain Rule: $\frac{d}{dx}[f(g(x))]=f'(g(x)) \cdot g'(x) = d/dx[f(u)]=f'(u)\cdot u'$ – Derivative of the function within in a function is the derivative of the outside function times the derivative of the inside function.	$\frac{d}{dX}((6X^3-2X)^2) = 8(27X^5-12X^3+X)$ $= 2(6X^3-2X)\left(\frac{d}{dX}(6X^3-2X)\right)$ $= 2(6X^3-2X)(6(3X^2)-2)$
	B. The General Power Rule: $\frac{d}{dx}[u^n]=nu^{n-1}\cdot u'$	

IV. Trigonometric and its Inverse

A. Trigonometric Functions:

2.2/3 1. *d/dx[sin x] = cos x*
2. *d/dx[cos x] = -sin x*
3. *d/dx[tan x] =* $\sec^2 x$
4. *d/dx[cot x] =* $-\csc^2 x$
5. *d/dx[sec x] = (secx)(tanx)*
6. *d/dx[csc x] = -(cscx)(cotx)*

5.6 B. Inverse Trigonometric Functions:

1. $d/dx[\sin^{-1} u]$ = *d/dx[arcsin u]* = $\frac{u'}{\sqrt{(1-u2)}}$
- Domain = [-1,1] Range = [$-\pi/2$,$\pi/2$]
2. $d/dx[\cos^{-1} u]$ = *d/dx[arccos u]* = $\frac{-u'}{\sqrt{(1-u2)}}$
- Domain = [-1,1] Range = [0, π]
3. $d/dx[\tan^{-1} u]$ = ***d/dx[arctan u] =*** $\frac{u'}{1+u^2}$

(most common)
- Domain = $(-\infty,\infty)$ Range = [$-\pi/2$,$\pi/2$]
4. $d/dx[\cot^{-1} u]$ = *d/dx[arccot u]* = $\frac{-u'}{1+u^2}$
- D = $(-\infty,\infty)$ R = [0, π]
5. $d/dx[\sec^{-1} u]$=*d/dx[arcsec u]*= $\frac{u'}{|u|\sqrt{u^2-1}}$
- D = $(-\infty,1]$ U $[1, \infty)$ R = $[0, \pi]$, , $y \neq \pi/2$
6. $d/dx[\csc^{-1} u]$=*d/dx[arccsc u]*= $\frac{-u'}{|u|\sqrt{u^2-1}}$
- D = $(-\infty,1]$ U $[1, \infty)$ R = $[-\pi/2 ,\pi/2]$, $y \neq 0$

V. Exponential and Logarithmic Functions

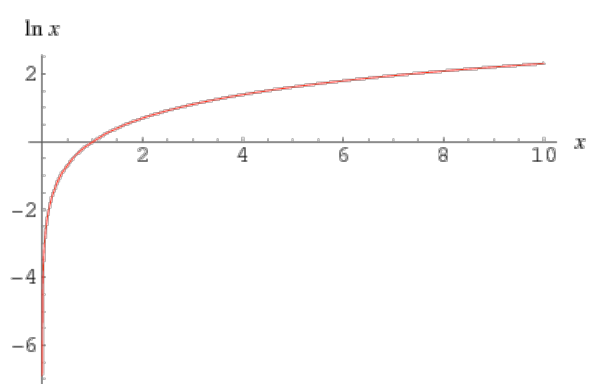
5.1 A. Natural Logarithmic function

1. Definition of Natural Log
   - $\int 1/t\, dt$ = *ln* x
2. Properties
   a. domain = $(0,\infty)$ and range = $(-\infty,\infty)$
   b. The function is continuous, increasing, and one-to one.
   c. The graph is concave downward.
3. More Properties
   a. ln (e) = 1
   b. ln(1) = 0
   c. ln(ab) = ln a + ln b
   d. ln(a/b) = ln a – ln b
   e. $\ln(a^n) = n \cdot \ln a$
4. Derivative
   a. *d/dx[ln x] = 1/x*
   b. *d/dx[ln u] = u'/u*

$$\frac{d}{dx}(\sin(2x)\cos(x)):$$
$$= -\sin(x)\sin(2x) + 2\cos(x)\cos(2x)$$
$$= \cos(x)\left(\frac{d}{dx}(\sin(2x))\right) + \sin(2x)\left(\frac{d}{dx}(\cos(x))\right)$$
$$= \cos(x)\left(\cos(2x)\left(\frac{d}{dx}(2x)\right)\right) + \sin(2x)\left(\frac{d}{dx}(\cos(x))\right)$$
$$= -\sin(x)\sin(2x) + 2\cos(x)\cos(2x)$$

$$\frac{d}{dx}\left(\tan^{-1}\left(x^2+3X\right)\right) = \frac{2x}{\left(x^2+3X\right)^2+1}$$
$$= \frac{\frac{d}{dx}\left(x^2+3X\right)}{\left(x^2+3X\right)^2+1}$$
$$= \frac{2x+30}{\left(x^2+3X\right)^2+1}$$

$$\frac{d}{dx}\left(\sec^{-1}\left(x^2-3\right)\right) = \frac{2x}{\left(x^2-3\right)^2\sqrt{1-\frac{1}{\left(x^2-3\right)^2}}}$$
$$= \frac{\frac{d}{dx}\left(x^2-3\right)}{\left(x^2-3\right)^2\sqrt{1-\frac{1}{\left(x^2-3\right)^2}}}$$
$$= \frac{\frac{d}{dx}\left(x^2\right)+0}{\left(x^2-3\right)^2\sqrt{1-\frac{1}{\left(x^2-3\right)^2}}}$$



$$\frac{d}{dX}\left(\log\left(3X^2-5X\right)\right) = \frac{5-6X}{5X-3X^2}$$
$$= \frac{\frac{d}{dX}\left(3X^2-5X\right)}{3X^2-5X}$$
$$= \frac{3(2X)-5}{3X^2-5X}$$

B. Exponential Functions

5.4 1. Definition

a. The inverse function of natural logarithmic function f(x) = ln x is called the natural exponential function ($f^{-1}(x) = e^x$)

b. $y = e^x$ if and only if $x = \ln y$

2. Properties

a. Domain = (-∞,∞) and range = (0,∞)

b. The function is continuous, increasing, and one-to-one on its entire domain.

c. The graph is concave upward.

d. limit of e^x as x approaches negative infinity is zero and limit of e^x as x approaches infinity is infinity.

3. Operations

a. $e^a e^b = e^{a+b}$

b. $e^a/e^b = e^{a-b}$

4. Derivative

a. *d/dx* $[e^x] = e^x$

b. *d/dx* $[e^u] = e^u \cdot u'$

C. Bases other than *e*

5.5 1. Definitions

a. if α is a positive real number (α ≠ 1) and x is any real number, then the exponential function to the base α is denoted by α^x and is defined by $\alpha^x = e^{(\ln\alpha)x}$.

(if α=1, then $y=1^x=1$ is a constant function)

b. if α is a positive real number (α ≠ 1) and x is any real number, then the logarithmic function to the base α is denoted by $\log_\alpha x$ and is defined as $\log_\alpha x = \ln x/\ln\alpha$

2. Properties of Inverse Functions

a. $y= \alpha^x$ if and only if $x=\log_\alpha y$

b. $\alpha^{\log_\alpha x}=x$, for x>0

c. $\log_\alpha \alpha^x =x$, for all x

3. Derivatives

a. *d/dx* $[\alpha^x]= (\ln\alpha)\, \alpha^x$

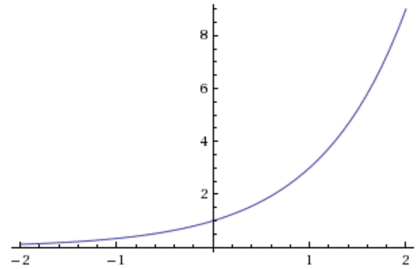
(*d/dx* $[\alpha^u]$ $(\ln\alpha)\, \alpha^u\, u'$)

b. *d/dx* $[\log_\alpha x] = \dfrac{1}{(\ln\alpha)\chi}$

(*d/dx* $[\log_\alpha u] = \dfrac{u'}{(\ln\alpha)u}$

2.5 **VI. Implicit Differentiation**

A. The strategy is useful - when you are unable to solve for y as a function of x



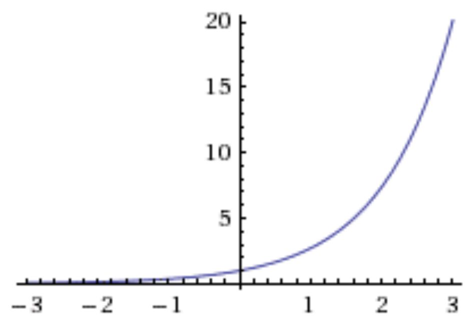
$$\frac{d}{dX}\left(e^{X^2-7X}\right) = e^{(X-7)X}\,(2X-7)$$

$$= e^{X^2-7X}\left(\frac{d}{dX}(X^2-7X)\right)$$

$$= e^{X^2-7X}\left(\frac{d}{dX}(X^2) - 7\left(\frac{d}{dX}(X)\right)\right)$$

$$= e^{X^2-7X}\,(2X-7)$$

Graph of 3^x



Log(x) is the natural logarithm

$$\frac{d}{dx}\left(\frac{3^{2x}}{x}\right) = \frac{9^x\,(x\log(9)-1)}{x^2}$$

$$= \frac{x\left(\frac{d}{dx}(3^{2x})\right) - 3^{2x}\left(\frac{d}{dx}(x)\right)}{x^2}$$

$$= \frac{x\left(3^{2x}\log(3)\left(\frac{d}{dx}(2x)\right)\right) - 3^{2x}\left(\frac{d}{dx}(x)\right)}{x^2}$$

$$= \frac{3^{2x}\,x\log(3)\left(2\left(\frac{d}{dx}(x)\right)\right) - 3^{2x}}{x^2}$$

$$= \frac{2\;1\;3^{2x}\,x\log(3) - 3^{2x}}{x^2}$$

B. Guidelines

1. Differentiate both sides of the equation **with respect to x**.
2. Collect all terms involving *dy/dx* on the left side of the equation and move all other terms to the right side of the equation.
3. Factor *dy/dx* out of the left side of the equation.
4. Solve for *dy/dx*.

10.3 **VII. Parametric Equations**

A. Definition

- If *f* and *g* are continuous functions of t on an interval *I*, then the equations *x=f(t)* and *y=g(t)* are called parametric equations and t is called the parameter.

B. Derivatives

1. $dy/dx = \frac{dy/dt}{dx/dt}$, $dx/dt \neq 0$
2. $d^2y/dx^2 = d/dx(dy/dx) = \frac{\mathrm{d/dt(dt/dx)}}{dx/dt}$

10.4 **VIII. Polar Coordinates and Equations**

A. To form the polar coordinate system in the plane, fix a point O, called the pole (or origin), and construct from O an initial ray called the polar axis.

1. r= directed distance from O to P
Θ= directed angle, counterclockwise from polar axis to segment OP
2. $r^2=x^2+y^2$
Tan θ = y/x, cos θ =x/r, sin θ =y/r

B. Coordinate Conversion.

The polar coordinates (r, Θ) of a point are related to the rectangular coordinates (x,y) of the point as follows.

1. x=r cos θ, y=r sin θ
2. tan θ = y/x

C. Derivative

1. If f is a differentiable function of Θ, then the slope of the tangent line to the graph of r=f(θ) at the point (r,θ) is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f(\theta)cos(\theta) + f'(\theta)sin(\theta)}{-f(\theta)sin(\theta) + f'(\theta)cos(\theta)}$$

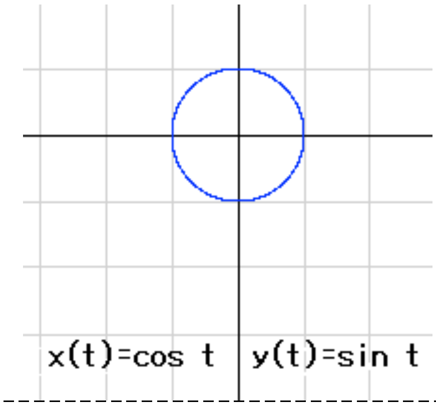
provided that dx/dθ ≠ 0 at (r,θ).

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$

$= dy/dx[3y^2+2y-5]-2x=0$

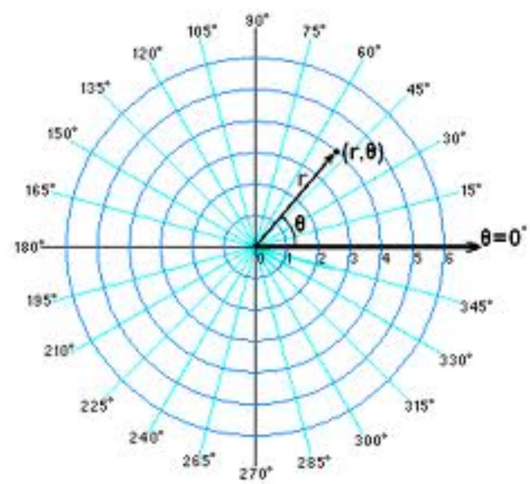
$= dy/dx[3y^2+2y-5]=2x$

$dy/dx = 2x/(3y^2+2y-5)$



Find the Derivative of y(t)=sin t and x(t) cos (t)

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t$$



Derivative of $x(\theta) = \cos\theta$ and $y(\theta) = \sin\theta$

$$\frac{dy}{dx} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

a. Solutions to dy/dθ = 0 yield horizontal tangents, provided that dx/dθ .
b. solutions to dx/dθ = 0 yield vertical tangents, provided that dy/dθ .
c. If dy/dϴ and dx/dθ are simultaneously 0, no conclusion can be drawn about tangent lines.

2. If f(α)=0 and f'(α)≠ 0, then the line θ= α is tangent at the pole to the graph of r=f(θ).

IIX. Vectors in the Plane

11.1 A. Definition

1. If ***v*** is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **component form of v** is given by
$\boldsymbol{v} = (v_1, v_2)$

2. The coordinates v_1 and v_2 are called the **components of *v***. If both the initial point and the terminal point lie at the origin, then **v** is called the **zero vector** and is denoted by **0** = <0, 0>

3. Two vectors $\boldsymbol{u} = \langle u_1, u_2 \rangle$ and $\boldsymbol{v} = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$

12.2 B. Differentiation of Vector-Valued Functions

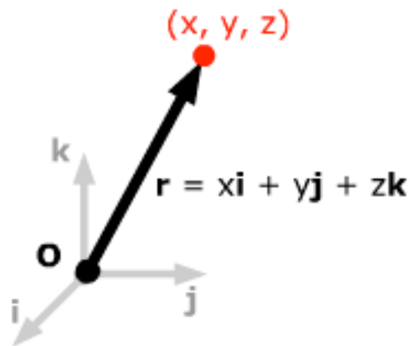
1. If **r**(t) = f(t)**i** + g(t)**j**, where *f* and *g* are differentiable functions of t, then
$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$

2. If **r**(t) = f(t)**i** + g(t)**j** + h(t)**k**, where *f*, *g* and *h* are differentiable functions of t, then
$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$

Derivative of

$x(\theta) = 3\sin 2\theta\cos\theta$ and $y(\theta) = 3\sin 2\theta\sin\theta$

$$\frac{dy}{dx} = \frac{6\cos 2\theta\sin\theta + 3\sin 2\theta\cos\theta}{6\cos 2\theta\cos\theta - 3\sin 2\theta\sin\theta}$$



$$r(t) = t^2 i + 4j$$

$$r'(t) = 2ti + 0j$$

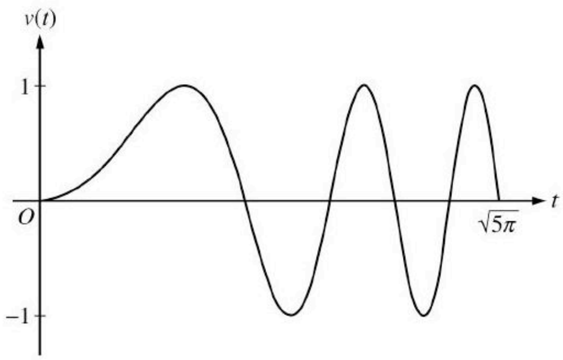
Practice Problems: Concept of a Derivative & Derivative at a point:

1. Find the following limits:

a. $\lim_{\Delta x \to 0} \dfrac{2(x+\Delta x)^3 - 5(x+\Delta x) - (2x^3 - 5x)}{\Delta x}$

b. $\lim_{x \to \pi/6} \dfrac{3\cos(x) - 3\cos(\pi/6)}{x - \pi/6}$

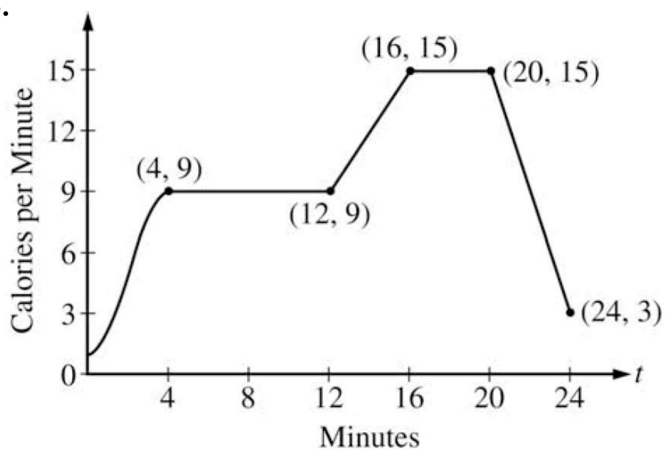
2.



A particle moves along the x-axis so that its velocity v at time $t \geq 0$ is given by $v(t) = \sin\left(t^2\right)$. The graph of v is shown above for $0 \leq t \leq \sqrt{5\pi}$. The position of the particle at time t is $x(t)$ and its position at time $t = 0$ is $x(0) = 5$.

(a) Find the acceleration of the particle at time $t = 3$.

3.



The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure, $f(t) = -(1/4)t^3 + (3/2)t^2 + 1$ for 0 ≤ t ≥ 4 and f is a piecewise linear for 4 ≤ t ≥ 24.

Find: (a) $f'(3)$ (b) $f'(22)$.

4. Find the equation of the tangent line to the graph of $f(x) = 5x^3 - 3x$, given $f(-1) = -2$, and $f'(-1) = 12$.

Solution Key: Concept of a Derivative & Derivative at a point:

1. (a) –Recognize that $\lim_{\Delta x \to 0} \frac{2(x+\Delta x)^3 - 5(x+\Delta x) - (2x^3 - 5x)}{\Delta x}$ is similar in form to

$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ which is equal to $f'(x)$.

By comparing the two equations we see that $f(x) = 2x^3 - 5x$ and therefore we can find that $f'(x) = 6x^2 - 5$.

Therefore, $\lim_{\Delta x \to 0} \frac{2(x+\Delta x)^3 - 5(x+\Delta x) - (2x^3 - 5x)}{\Delta x} = 6x^2 - 5$

(b) –Recognize that $\lim_{x \to \pi/6} \frac{3\cos(x) - 3\cos(\pi/6)}{x - \pi/6}$ is similar in form to $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ which is equal to $f'(c)$.

By comparing the two equations we see that $f(x) = 3\cos(x)$ and therefore we can find that $f'(x) = -3\sin(x)$.

By plugging in $c = \pi/6$ we get $f'(c) = -3/2$

Therefore, $\lim_{x \to \pi/6} \frac{3\cos(x) - 3\cos(\pi/6)}{x - \pi/6} = -3/2$

2. –Since $a(t) = v'(t)$ where ($v(t) = x(t)$) we need to find the derivative $x'(3)$

$$x(t) = \sin(t^2)$$
$$x'(t) = 2t\cos(t^2)$$
$$x'(3) = 6\cos(9) \approx -5.467$$

3. (a) –For t=3, which falls in the interval 0 ≤ t ≥ 4, we use $f(t) = -(1/4)t^3 + (3/2)t^2 + 1$

$$f(t) = -(1/4)t^3 + (3/2)t^2 + 1$$
$$f'(t) = -(3/4)t^2 + 3t$$
$$f'(3) = 2.25$$

(b) –For t=22, which falls on the interval4 ≤ t ≥ 24, the graph is a piecewise funtion. We can estimate the derivative from the slope of the line.

Slope= $\dfrac{15-3}{20-24} = \dfrac{12}{-4} = -3$

Therefore, $f'(22) = -3$

4. –Using tangent line approximation we know that the equation of a tangent line is:

$$y = f(c) + f'(c)(x-c)$$

where:

$$c = -1$$
$$f(c) = -2$$
$$f'(c) = 12$$

Therefore, the equation is:

$$y = -2 + 12(x-(-1)) = 12(x+1) - 2$$
$$y = 12x + 10$$

Calc Sample Problems

*"derivative as a function" and "second derivative"

Ch - 3.2 The Mean Value Theorem

For problems #1-2, find the value of c in the given interval which satisfies the conclusion of the mean value theorem

1) $f(x) = x^4 - 16x^2 + 2, -1 \leq x \leq 3$
2) $f(x) = x \cos(\sqrt{x}),\ 0 \leq x \leq 50$
3) The height of an object t seconds after it is dropped from a height of 500 meters is $s(t) = -4.9t^2+500$. Use the Mean Value Theorem to find the time that the instantaneous velocity = the average velocity of -14.7 m/s. (p177 #51b)

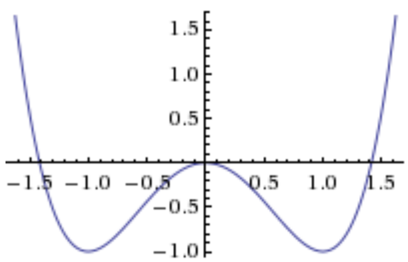
Ch - 6.2 Equations Involving Derivatives

Write the differential equation that models the verbal statement or vice versa (p 418 # 11-14)

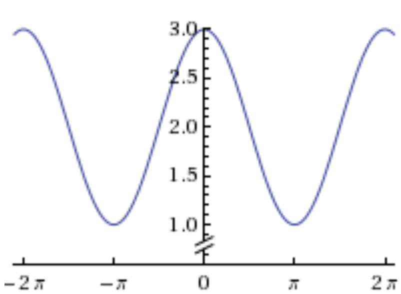
4) The rate of change of *Q* with respect to *t* is inversely proportional to the square of *t*
5) The rate of change of *P* with respect to t is proportional to 10 – *t*
6) $N'(s) = k(250 - s)$
7) $y'(x) = \dfrac{k}{x(L-y)}$

Ch - 3.3 Relationship between the behavior of f and the sign of f′

8) Given the graph of f′(x), describe the behavior of f

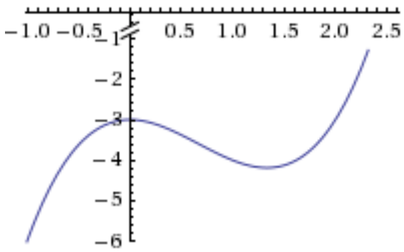


9) Given the graph of f(x), describe the sign of f in the interval (-2Π , 2 Π)

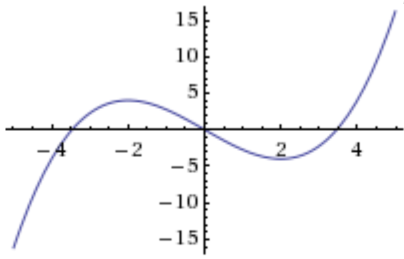


Ch - 3.3 Corresponding Characteristics of f(x) and f'(x)

10) Given the graph of $f(x) = x^3 - 2x^2 - 3$ what can you deduce about the graph of $f'(x)$?



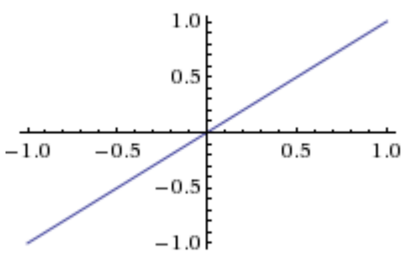
11) Given the graph of $f'(x) = \frac{x^3}{4} - 3x$ what can you deduce about the graph of $f(x)$?



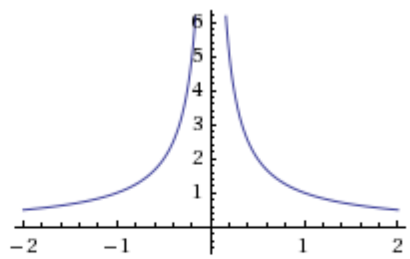
Ch - 3.4 Characteristics of the graphs of f(x), f'(x) and f''(x)

In #12, 13, the graph of f(x) is shown. Sketch f(x), f'(x) and f''(x) on the same coordinate plane. (p196 # 49,52)

12)



13)



Ch - 3.4 Relationship between the concavity of f(x) and the sign of f''(x) and points of inflection

14) T/F : If f'(c) > 0, then f is concave upward at x = c

15) Determine the open intervals in which the function $f(x) = \frac{x^2 + 1}{x^2 - 4}$ is concave upward or downward.

16) Show that the point of inflection of $f(x) = x(x-6)^2$ lies midway between the relative extrema of f.

Calc Sample Problem Solutions

* "derivative as a function" and "second derivative"

1) $f(x) = x^4 - 16x^2 + 2 \quad [-1, 3]$

- According to the Mean Value Theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{so if } \begin{matrix} a=-1 \\ b=3 \end{matrix}, \quad f'(c) = \frac{[3^4 - 16(3)^2 + 2] - [(-1)^4 - 16(-1)^2 + 2]}{3 - (-1)}$$

- $f'(c) = -12$

- if $f'(x) = 4x^3 - 32x$

then $-12 = 4c^3 - 32c$

$$\boxed{c = -3, 2.618, .382}$$

2) $f(x) = x \cos(\sqrt{x})$

- According to the MVT,

if $\begin{matrix} a=0 \\ b=50 \end{matrix}$ $\quad f'(c) = \dfrac{50 \cos\sqrt{50}}{50}$

- $f'(c) = \cos\sqrt{50}$
- if $f'(x) = \cos(\sqrt{x}) - x\sin(\sqrt{x})$

then $\cos\sqrt{50} = \cos(\sqrt{c}) - c\sin(\sqrt{c})$

$c =$

3) $s(t) = -4.9t^2 + 500$, velocity $= -14.7$ m/sec

- if $\begin{matrix} a=0 \\ b=3 \end{matrix}$, $\quad f'(c) = \dfrac{(-4.9(3)^2 + 500) - 500}{3}$

- $s'(t) = -14.7$
- if $s'(t) = -9.8t$

then $-14.7 = -9.8c$

$c = 1.5$

- The instantaneous velocity will equal -14.7 m/sec when $t = 1.5$ sec.

4) By the definition of "inverse proportionality", if $f(x)$ is inversely proportional to P, $f(x) = \frac{k}{P}$

So

$$Q'(t) = \frac{k}{t^2}$$

5) If $f(x) \propto P$, $f(x) = kP$

So

$$P'(t) = k(10-t)$$

6) By observing the function, we determine that the relationship is directly proportional

So

The rate of change of N with respect to s is proportional to $250 - s$.

7) We can determine that the relationship varies jointly

So

The rate of change of y with respect to x varies jointly as x and $(L-y)$.

8) According to the rule given in Ch 3.3, when $f'(x)$ is positive, $f(x)$ is increasing and when $f'(x)$ is negative, $f(x)$ is decreasing. $\therefore$ we can conclude that $f(x)$ is increasing in the intervals $(-\infty, \approx -1.5)$ and $(\approx 1.5, \infty)$. Likewise, $f(x)$ is decreasing in the interval $(\approx -1.5, \approx 1.5)$ except at $x = 0$ where $f(x)$ has no slope.

9) By the same reasoning employed in #8, we can conclude that $f'(x)$ is positive in the intervals $(-\pi, 0)$ and $(\pi, 2\pi)$. Likewise, $f'(x)$ is negative in the intervals $(-2\pi, -\pi)$ and $(0, \pi)$.

10) • Since $f(x)$ is increasing in the intervals $(-\infty, 0)$ and $(\approx 1.5, \infty)$, $f'(x)$ is positive in those intervals

• Since $f(x)$ is decreasing in the interval $(0, \approx 1.5)$, $f'(x)$ is negative in that interval

• Since $f(x)$ has no slope at $x = 0$ and $x \approx 1.5$, $f'(x) = 0$ at those points

• Since $f(x)$ has a relative maximum at $x = 0$, $f'(x)$ crosses the x-axis in a negative direction at that point

• Since $f(x)$ has a relative minimum at $x \approx 1.5$, $f'(x)$ crosses the x-axis in a positive direction at that point

11) • Since $f'(x)$ is positive in the intervals $(\approx -4, 0)$ and $(\approx 4, \infty)$, $f(x)$ has a positive slope in those intervals

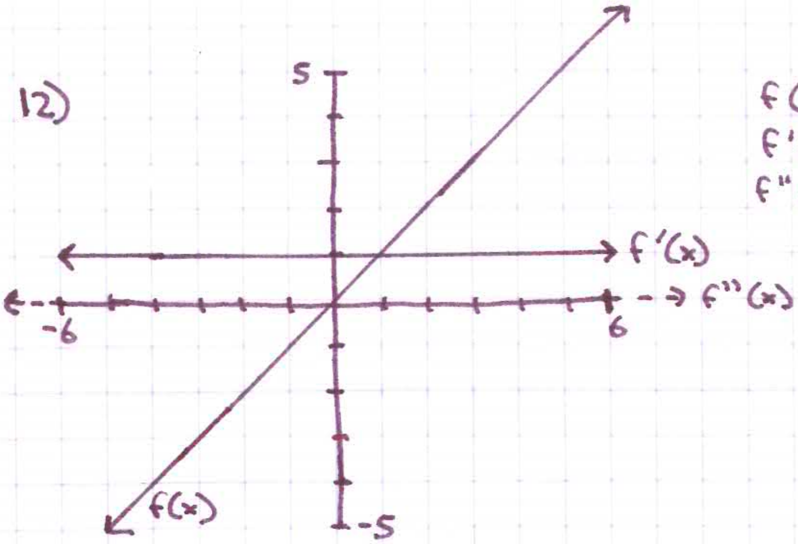
• Since $f'(x)$ is negative in the intervals $(-\infty, \approx -4)$ and $(0, \approx 4)$, $f(x)$ has a negative slope in those intervals

• Since $f'(x) = 0$ at $x \approx -4$ and $x \approx 4$, $f(x)$ must have no slope at those points

• Since $f'(x)$ crosses the x-axis in a positive direction at $x \approx \pm 4$, $f(x)$ must have a relative minimum at those points

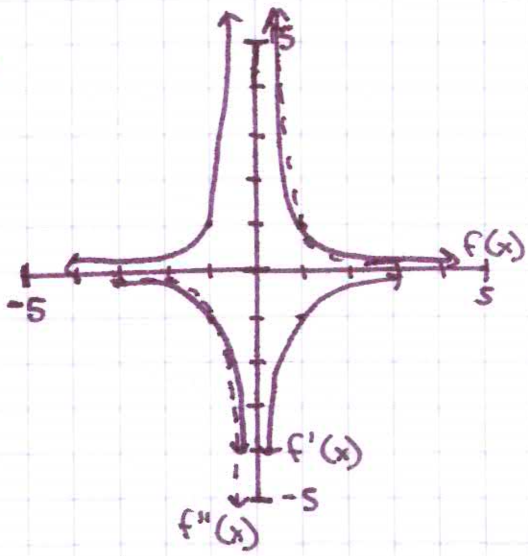
• Since $f'(x)$ crosses the x-axis in a negative direction at $x \approx 0$, $f(x)$ must have a relative maximum at those points

12)



$f(x) = x$
$f'(x) = 1$
$f''(x) = 0$

13)



$f(x) = \frac{1}{x}$
$f'(x) = -x^{-2}$ or $-\frac{1}{x^2}$
$f''(x) = 2x^{-3}$ or $\frac{2}{x^3}$

14) False. If $f'(c) > 0$, one can only infer that f has a positive slope at that point. Positive concavity occurs only when $f''(c) > 0$.

15) • To determine the second derivative of $f(x)$, we differentiate twice

$$f(x) = \frac{x^2+1}{x^2-4}$$

$$f'(x) = \frac{(x^2-4)(2x) - (x^2+1)(2x)}{(x^2-4)^2} = \frac{-10x}{(x^2-4)^2}$$

$$f''(x) = \frac{(x^2-4)^2(-10) - (-10x)(2)(x^2-4)(2x)}{(x^2-4)^4} = \frac{10(3x^2+4)}{(x^2-4)^3}$$

• Since $f''(x)$ is never equal to zero, we determine the values for which $f''(x)$ is undefined

$$(x^2-4)^3 = 0$$
$$x^2-4 = 0$$
$$x = \pm 2$$

- Test for concavity in the intervals determined by your PIP's.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	+	-	+
Conclusion	concave up	concave down	concave up

- $f(x)$ is concave upward in the intervals $(-\infty, -2)$ and $(2, \infty)$. It is concave downward in the interval $(-2, 2)$.

16)

- First, we use the first derivative to locate the relative extrema

$$f(x) = x(x-6)^2$$
$$f'(x) = (x-6)^2 + (x)2(x-6) \qquad \text{Product Rule}$$
$$f'(x) = 3x^2 - 24x + 36$$
$$x = 6, 2 \quad \Rightarrow \text{The relative extrema occur here}$$

- Now, we locate the inflection point using the second derivative

$$f'(x) = 3x^2 - 24x + 36$$
$$f''(x) = 6x - 24$$
$$x = 4$$

- Since $|6-4| = 2$ and $|2-4| = 2$, the inflection point is equidistant from both extrema.

Sample Problems on Computation of Derivatives

Sections

1) Basic Rules + Quotient Rule- 2.2/3

2) Basic Trig Rules -2.2/3

3) Chain Rule- 2.4

4) Implicit Differentiation -2.5

5+6) ln functions-5,1

7) Exponential function -5.4

8) Bases other than e – 5.5

9+10) Inverse trigonometric functions – 5.6

11)Parametric Functions- 10.3

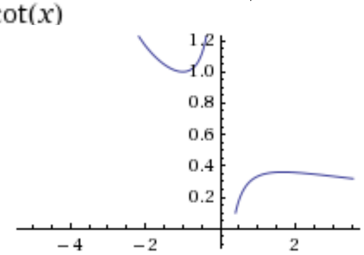
12) Polar Equations – 10.4

13) Vector Valued Functions – 12.2

14+15) Combination of Derivatives

1) Find and equation of the tangent line to the graph $f(x) = \dfrac{3 - \frac{1}{x}}{x+5}$ at (-1,1)

2) differentiate both forms of: $y = \dfrac{1-\cos(x)}{\sin(x)} = \csc(x) - \cot(x)$



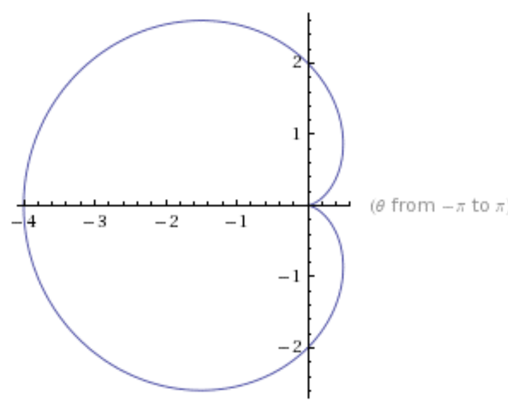
3) Find the derivative of $f(x) = \left(\dfrac{3x-1}{x^2+3}\right)^2$

4) Given $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$

5) Find the derivative of $y = \dfrac{(x-2)^2}{\sqrt{x^2+1}}$, $x \neq 2$ (hint: use Natural Logarithm)

6) $\dfrac{d}{dx}[x \ln x] = ?$

7) $\dfrac{d}{dx}\left(e^{-3/x}\right) = ?$



8) Find the derivative of $y = \log_{10} \cos x$

9) $\dfrac{d}{dx}[\arctan(3x)] = ?$

10) Differentiate $y = \arcsin x + x\sqrt{1-x^2}$

11) Find the slope of $x = \sqrt{t}$ and $y = \dfrac{1}{4}(t^2 - 4)$ at (2,3)

12) Find the horizontal and vertical tangents to the graph of $r = 2(1 - \cos\theta)$

13) Find the derivative of vector-valued function $r(t) = \dfrac{1}{t}i + \ln t\, j + e^{2t}k$, $0 \leq \theta \leq \pi$

14) Find the derivative of $y = \dfrac{2x}{\sqrt{x+1}}$

15) Find the derivative of $y - \csc^3(\sqrt{x})$

Solutions to the Key Problems

#1. $f(x) = \dfrac{3 - 1/x}{x+5}$

$= \dfrac{x}{x}\left(\dfrac{3-\frac{1}{x}}{x+5}\right)$

$= \dfrac{3x-1}{x^2+5x}$

$f'(x) = \dfrac{(x^2+5x)(3)-(3x-1)(2x+5)}{(x^2+5x)^2}$

$= \dfrac{(3x^2+15x)-(6x^2+13x-5)}{(x^2+5x)^2}$

$= \dfrac{-3x^2+2x+5}{(x^2+5x)^2}$

$f'(-1) = 0$

$y - 1 = 0(x+1)$

$\boxed{y = 1}$

#2 $y = \dfrac{1-\cos x}{\sin x}$

$y' = \dfrac{(\sin x)(\sin x) - (1-\cos x)(\cos x)}{\sin^2 x}$

$= \dfrac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x}$

$= \dfrac{1-\cos x}{\sin^2 x} = \dfrac{1}{\sin^2 x} - \left(\dfrac{1}{\sin x}\right)\left(\dfrac{\cos x}{\sin x}\right) = \csc^2 x - \csc x \cot x$

$y = \csc x - \cot x$

$y' = -\csc x \cot x + \csc^2 x$

#3 $y = \left(\dfrac{3x-1}{x^2+3}\right)^2$

$y' = 2\left(\dfrac{3x-1}{x^2+3}\right)\left[\dfrac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right]$

$= \dfrac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3}$

$= \dfrac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$

#4 $x^2+y^2=25$

$2x+2y\frac{dy}{dx}=0$

$2y\frac{dy}{dx}=-2x$

$\frac{dy}{dx}=\frac{-2x}{2y}=\boxed{\frac{-x}{y}}$

$$-\frac{(y)(1)-(x)(dy/dx)}{y^2}$$

$$=-\frac{y-(x)(-x/y)}{y^2}$$ ← substitution

$$=-\frac{y^2+x^2}{y^3}$$ ← substitution

$$=\boxed{-\frac{25}{y^3}}$$

#5 $y=\frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x\neq 2$

$\ln y=\ln\frac{(x-2)^2}{\sqrt{x^2+1}}$

$\ln y=2\ln(x-2)-\frac{1}{2}\ln(x^2+1)$

$\frac{y'}{y}=2\left(\frac{1}{x-2}\right)-\frac{1}{2}\left(\frac{2x}{x^2+1}\right)$

$=\frac{2}{x-2}-\frac{x}{x^2+1}.$

$y'=y\left(\frac{2}{x-2}-\frac{x}{x^2+1}\right)$

$=\frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right]$ substitution.

$=\boxed{\frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}}}$

#6 $\frac{d}{dx}[x\ln x]=x\left(\frac{d}{dx}[\ln x]\right)+(\ln x)\left[\frac{d}{dx}[x]\right]$

$=x\left(\frac{1}{x}\right)+(\ln x)(1)=\boxed{1+\ln x}$

#7 $\frac{d}{dx}(e^{-3/x})=e^u\left(\frac{du}{dx}\right)=\left(\frac{3}{x^2}\right)e^{-3/x}=\boxed{\frac{3e^{-3/x}}{x^2}}$

#8 $y=\log_{10}\cos x$

$y'=\frac{-\sin x}{(\ln 10)\cos x}=\boxed{-\frac{1}{\ln 10}\tan x}$

#9. $\frac{d}{dx}[\arctan(3x)] = \frac{3}{1+(3x)^2} = \boxed{\frac{3}{1+9x^2}}$

#10. $y = \arcsin x + x\sqrt{1-x^2}$

$$y' = \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x) + \sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$

$$= \sqrt{1-x^2} + \sqrt{1-x^2}$$

$$= \boxed{2\sqrt{1-x^2}}$$

#11. $x = \sqrt{t} \qquad y = \frac{1}{4}(t^2-4), \quad t \geq 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1/2)t}{(1/2)t^{-1/2}} = t^{3/2}$$

$2 = \sqrt{t}$
$4 = t$
$3 = \frac{1}{4}(t^2-4)$
$4 = t$

At $(x,y) = (2,3)$, it follows that $t = 4$

$$\frac{dy}{dx} = (4)^{3/2} = \boxed{8}$$

#12 $x = r\cos\theta \qquad y = r\sin\theta \quad , r = \sin\theta \quad 0 \leq \theta \leq \pi.$

$x = \sin\theta\cos\theta \qquad y = \sin\theta\sin\theta = \sin^2\theta$

$$\frac{dx}{d\theta} = \cos\theta(\cos\theta) + \sin\theta(-\sin\theta) = \cos^2\theta - \sin^2\theta$$

$$= \cos 2\theta = 0 \implies \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\frac{dy}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta = 0 \implies \theta = 0, \frac{\pi}{2}.$$

So, the graph has vertical tangent lines at $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$ and $\left(\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$, and it has horizontal tangent lines at $(0,0)$ and $\left(1, \frac{\pi}{2}\right)$.

#13 $r(t) = \frac{1}{t} i + \ln t\, j + e^{2t} K$

$$\boxed{r'(t) = \frac{-1}{t^2} i + \frac{1}{t} j + 2e^{2t} K}$$

#14. $y = \frac{2x}{\sqrt{x+1}}$

$$y' = \frac{2(\sqrt{x+1}) - 2x\left(\frac{1}{2}(x+1)^{-\frac{1}{2}}\right)}{(\sqrt{x+1})^2}$$

$$= \frac{(x+1)^{-\frac{1}{2}}\left[2(x+1) - x\right]}{(x+1)}$$

$$= \frac{2x+2-x}{\sqrt{x+1}\,(x+1)} = \boxed{\frac{x+2}{(x+1)^{3/2}}}$$

#15. $y = \csc^3(\sqrt{x})$

$$y' = 3\csc^2(\sqrt{x})\left(-\csc\sqrt{x}\cot\sqrt{x}\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= \boxed{\frac{-3\csc^3\sqrt{x}\cot\sqrt{x}}{2\sqrt{x}}}$$